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Résumé: Les jeux d'arrêt sont des jeux séquentiels où, à chaque étape, chacun des joueurs peut décider d'arrêter ou de continuer. Le jeu s'arrête dès lors que l'un au moins des joueurs décide de s'arrêter. Le paiement reçu alors par les joueurs dépend de l'ensemble des joueurs qui ont choisi de s'arrêter à cette date. Si le jeu ne s'arrête jamais, le paiement est nul.
Nous étudions un jeu à quatre joueurs. Dans ce jeu, les équilibres les plus simples sont périodiques de période deux. Par ailleurs, nous utilisons des outils géométriques pour montrer que les techniques utilisées pour les jeux à trois joueurs ne peuvent être adaptées au cas général.

Abstract: Quitting games are I-player sequential games in which, at any stage, each player has the choice between continuing and quitting. The game ends as soon as at least one player chooses to quit; player i then receives a payoff, which depends on the set of players that did choose to quit. If the game never ends, the payoff to each player is zero.
We study a four-player game, where the simplest equilibrium profile is cyclic with period two. We also use geometrical arguments to show why techniques that work for 3-player games cannot be applied in general.

Mots clés : Jeux d'arrêt, Jeux stochastiques, Equilibre

Key Words : Quitting games, Stochastic Games, Equilibrium

Classification AMS: 91A15, 91A55, 91A06

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1 Introduction

Quitting games are I-player sequential games in which, at any stage, each player has the choice between continuing and quitting. We denote the two actions of player i by $\{q^i, c^i\}$. The game ends as soon as at least one player chooses to quit; player i then receives a payoff r^i_S, which depends on the set S of players that did choose to quit. If the game never ends, the payoff to each player is 0.

In such a game, a strategy of player i is a sequence $\mathsf{x}^i = (x^i_n)_{n\geq 0}$, where x^i_n is the probability that player i continues at stage n, provided the game has not terminated before. Such a strategy is *stationary* if x^i_n is independent of i. We denote by a^i_n the action played by player i at stage n, and denote by $t = \inf\{n \geq 1, \mathsf{a}^i_n = q^i \quad \text{for some player} i \in I\}$ the stage in which the game terminates, and by S_t the set of players that choose to quit at that stage. Given a profile x of strategies, the expected payoff to player i is

$$\gamma^i(\mathsf{x}) = \mathsf{E}_{\mathsf{x}}\left[r^i_{S_t} \mathbf{1}_{t<+\infty} \right],$$

where E_{x} stands for the expectation with respect to the probability distribution induced by x over the set of plays.

It is not known whether quitting games have ε-equilibrium. Quitting games therefore form an intriguing class of stochastic games. We recall briefly existing results before presenting the contribution of this note.

In the case of two players, stationary ε-equilibria do exist. A three-player example was devised by Flesch, Thuijsman and Vrieze (1997), where ε-equilibrium strategies are more complex - they have a cyclic structure, and the length of the cycle is at least 3. However, in this example, there are equilibrium payoffs in the convex hull of the vectors $r_{\{i\}} \in \mathsf{R}^I$, $i \in I$. These payoffs can be obtained using a profile x that plays in any stage a perturbation of $(c^i)_{i\in I}$. Therefore, it left open the possibility of finding ε-equilibrium profiles, by means of analyzing the limit behavior of stationary equilibria of discounted games, letting the discount factor go to zero. Indeed, such an analysis was provided by Solan (1999), for the more general class of three-player games with absorbing states. These results are discussed in Section 2.

The purpose of this note is twofold. First, we explain why the techniques that work for 4-player games fail in general. Second, we provide a 4-player example, where all the ε-equilibrium payoffs involve some kind of cyclic behavior, in which the probability of quitting in any stage is bounded away from zero. The main consequence is that all the known tools for proving the existence of equilibrium payoffs in stochastic games (see, e.g., Tuijsman and Vrieze (1989), Solan (2000), Vieille (2000a,2000b)) seem likely to fail to yield any result in general I-player quitting games. In a companion paper (Solan and Vieille (2001)) we introduced new tools and provided sufficient conditions under which quitting games admit an equilibrium payoffs.

2 Two- and three-player quitting games

The purpose of this section is to discuss the case where the number of players does not exceed 3. We shall hint that such games always have ε-equilibria of a simple form.

We first introduce a few notations. Given a profile x, and a stage $n \in N$, we denote by $\mathsf{x}_n = (x_n, x_{n+1}, ...)$ the profile induced by x in the subgame starting from stage n. We let c denote the profile of actions (c^i), and by c^i the pure stationary strategy that plays repeatedly c^i. When convenient, we shall not distinguish between a stationary strategy $\mathsf{x}^i = (x^i, ..., x^i)$ and the probability $x^i \in [0, 1]$.

2.1 Two-player quitting games

For notational convenience, we represent a two-player quitting game as

	c^2	q^2
c^1		(b_1, b_2)
q^1	(a_1, a_2)	(d_1, d_2)

If there is a pure stationary equilibrium we are done. Otherwise either $a_1 > 0$ or $b_2 > 0$ (otherwise $(\mathsf{c}^1, \mathsf{c}^2)$ is an equilibrium). Assume w.l.o.g. that $a_1 > 0$. Then $a_2 < d_2$, which implies that $d_1 < b_1$, which implies that $b_2 < 0$.

If $a_2 \geq b_2$ then the stationary strategy $(\mathsf{x}^1, \mathsf{c}^2)$ is an ε-equilibrium, where $x^1_n = (1 - \eta)c^i + \eta q^i$, and $\eta \in (0, 1)$ is sufficiently small.

Of $a_2 < b_2$ then the stationary strategy $(\mathsf{x}^1, \mathsf{q}^2)$ is an ε-equilibrium, where x^1 is defined as above.

Therefore, two-player quitting games always have stationary ε-equilibrium. Note that equilibria need not exist, as e.g. in the zero-sum game

	c^2	q^2
c^1		$(1, -1)$
q^1	$(1, -1)$	$(0, 0)$

2.2 Three-player quitting games

Flesch et *al.* (1997) exhibited a three-player quitting game with no stationary ε-equilibrium. Solan (1999) proved the existence of ε-equilibria for the more general class of three-player absorbing games. When specialized to quitting games, Solan's proof yields a profile $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ that is either stationary or such that $\|x_n - c\| < \varepsilon$ for each $n \in \mathbb{N}$. The proof is based on a vanishing discounting argument.

We offer here a geometric argument that is specific to the case of quitting games, in the hope of providing a better understanding of why the generalization to more-than-three-player games does not hold. However, it is not our intention to provide a detailed proof. In particular, we shall only deal with the case where $r^i_{\{i\}} > 0$ for each $i \in \mathcal{N}$. The discussion of the other cases is somewhat similar.

We normalize the payoffs to have $r^i_{\{i\}} = 1$ for each i. We organize the discussion according to the configuration of payoffs. The different cases are exhaustive, but not mutually exclusive. All strategies are stationary unless explicitly specified.

For every $\varepsilon \in (0,1]$ define $T_\varepsilon = \{x \in [0,1]^3 \mid \sum_{i=1}^3 x^i = \varepsilon$, and $\Delta_\varepsilon = \{x \in [0,1]^3 \mid \sum_{i=1}^3 x^i \geq \varepsilon$.

Case 0: There exists $\varepsilon \in (0,1)$ such that for every profile $x \in T_\varepsilon$ there is at least one player i whose unique best reply to x is q^i.

We prove that the game has a stationary equilibrium. The proof is based on a standard fixed-point argument, applied to the best-replies of a constrained game.

For every $x \in T_\varepsilon$ let $I_x \subseteq I$ be the set of players i such that $\gamma^i(x^{-i}, q^i) - \gamma^i(x) > 0$. The assumption tells us that I_x is not empty for every $x \in T_\varepsilon$. Since $\gamma^i(x)$ and $\gamma^i(x^{-i}, q^i)$ are continuous over the compact set T_ε, $\rho = \min_{x\in T_\varepsilon} \max_{i\in I_x} \{\gamma^i(x^{-i}, q^i) - \gamma^i(x)\} > 0$.

It follows that there is $\varepsilon_1 > \varepsilon$ such that for every $x \in T_{\varepsilon_1}$ there is a player i such that $\gamma^i(x^{-i}, q^i) - \gamma^i(x) > \rho/2$. Define a continuous function $f : \Delta_\varepsilon \to \Delta_\varepsilon$ by

$$f^i(x) = \begin{cases} x^i + (\varepsilon_1 - \varepsilon)(\gamma^i(x^{-i}, q^i) - \gamma^i(x)) & \gamma^i(x^{-i}, q^i) \geq \gamma^i(x) \\ x^i + \frac{\rho}{2}(\varepsilon_1 - \varepsilon)(\gamma^i(x^{-i}, q^i) - \gamma^i(x)) & \gamma^i(x^{-i}, q^i) < \gamma^i(x). \end{cases}$$

Since f is continuous, it has a fixed point in Δ_ε, which is a stationary equilibrium.

Case 1: $r^2_{\{1\}}, r^3_{\{1\}} \geq 1$.

In that case, both players 2 and 3 are at worst indifferent between quitting alone or waiting for player 1 to quit. The stationary profile $((1-\eta)\mathsf{c}^1+\eta\mathsf{q}^1,\mathsf{c}^2,\mathsf{c}^3)$ is an ε-equilibrium, provided η is sufficiently small.

This analysis remains valid when the roles of the players are permuted.

Case 2 : There is no convex combination $\alpha_1 r_{\{1\}}+\alpha_2 r_{\{2\}}+\alpha_3 r_{\{3\}}$ of the three vectors $(r_{\{1\}},r_{\{2\}},r_{\{3\}})$ such that $\alpha_1 r_{\{1\}}+\alpha_2 r_{\{2\}}+\alpha_3 r_{\{3\}}\geq(1,1,1)$.

By continuity, there is $\rho>0$ such that in every convex combination of $r_{\{1\}}$, $r_{\{2\}}$ and $r_{\{3\}}$, at least one player receives at most $1-\rho$. It follows that for $\varepsilon>0$ sufficiently small, the assumption of Case 0 holds. In particular, there is a stationary equilibrium.

Case 3 : $r^1_{\{2\}},r^1_{\{3\}}<1$.

One can easily verify that the assumption of Case 1 or Case 2 is satisfied.

Case 4: There is a convex combination $\alpha_1 r_{\{1\}}+\alpha_2 r_{\{2\}}+\alpha_3 r_{\{3\}}$ of the three vectors $(r_{\{1\}},r_{\{2\}},r_{\{3\}})$ such that $\alpha_1 r_{\{1\}}+\alpha_2 r_{\{2\}}+\alpha_3 r_{\{3\}}=(1,1,1)$.

The stationary profile $((1-\eta\alpha_1)\mathsf{c}^1+\eta\alpha_1\mathsf{q}^1,(1-\eta\alpha_2)\mathsf{c}^2+\eta\alpha_2\mathsf{q}^2,(1-\eta\alpha_3)\mathsf{c}^3+\eta\alpha_3\mathsf{q}^3)$ is an ε-equilibrium, provided η is sufficiently small.

We next introduce a convenient notational convention. For $i\neq j$, we shall write $r^i_{\{j\}}=$ '$+$' if $r^i_{\{j\}}\geq 1$ and $r^i_{\{j\}}=$ '$-$' if $r^i_{\{j\}}<1$. If none of the assumptions of Case 1 and Case 3 are satisfied, the triple $(r_{\{1\}},r_{\{2\}},r_{\{3\}})\in\mathbf{R}^9$ is either of the form $((1,+,-),(-,1,+),(+,-,1))$ or $((1,-,+),(+,1,-),(-,+,1))$. Each of these two situations is reducible to the other by permuting two players. We will proceed under the assumption that

$$(r_{\{1\}},r_{\{2\}},r_{\{3\}})\text{ is of the form }((1,+,-),(-,1,+),(+,-,1)).$$

Case 5: There is a convex combination $\alpha_1 r_{\{1\}}+\alpha_2 r_{\{2\}}+\alpha_3 r_{\{3\}}$ of the three vectors $(r_{\{1\}},r_{\{2\}},r_{\{3\}})$ such that $\alpha_1 r_{\{1\}}+\alpha_2 r_{\{2\}}+\alpha_3 r_{\{3\}}\geq(1,1,1)$.

The set of such $(\alpha_1,\alpha_2,\alpha_3)$ is the intersection of three hyperplanes, each involves two constraints, hence it is either a singleton or a triangle. If it is a singleton the assumption of Case 4 is satisfied.

We proceed with the case where the set is a triangle. The vertices are labelled A,B,C in such a way that players 1 and 3 (resp. 1 and 2, 2 and 3) get a payoff equal to one under the convex combination $A=(\alpha^A_1,\alpha^A_2,\alpha^A_3)$ (resp. B,C) (see the figure).

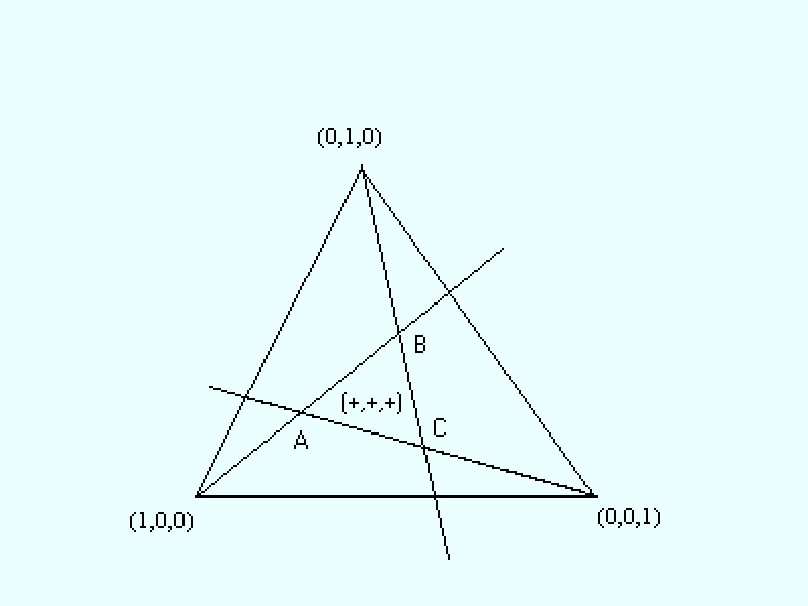

Figure 1

We next write A (resp. B, C) as a convex combination of $(1,0,0)$ and B (resp. of $(0,1,0)$ and C, of $(0,0,1)$ and A):

$$\begin{aligned} A &= \beta_1(1,0,0) + (1-\beta_1)B \\ B &= \beta_2(0,1,0) + (1-\beta_2)C \\ C &= \beta_3(0,0,1) + (1-\beta_3)A. \end{aligned}$$

Fix $M \in \mathsf{N}$, large enough. Define a non-stationary profile σ as follows. Players 1, 2 and 3 (in that order) alternate indefinitely as follows. During M stages, player i quits with probability $\frac{\beta_i}{M}$ (while the other two players continue). Depending on who starts first, the payoff induced by σ is close to the payoff associated with the convex combination A, B or C respectively. Moreover, the profile σ is an ε-equilibrium of the quitting game.

Therefore, for three-player quitting games, there always exists a stationary ε-equilibrium or an ε-equilibrium $\mathsf{x} = (x_n)_{n \in \mathsf{N}}$, such that $\|x_n - c\| < \varepsilon$ for each n.

2.3 Discussion

This geometric construction may help to understand why in general there need not be neither a stationary ε-equilibrium nor an equilibrium payoff in the convex hull of $\{r_{\{i\}}\}_{i \in I}$. Assume for simplicity that $r^i_{\{i\}} = 1$ for each player i.

Consider for a moment a 3-player game. From each point $u \in [1, \infty)^3$, and every i such that $u^i = 1$, draw a small arrow in the direction $u - r_{\{i\}}$ (the number of arrows from each u can be 0,1,2 or 3). If there is a fixed point u (that is, $u = r_{\{i\}}$ for some player i), then there is a stationary ε-equilibrium, that corresponds to Case 1 above. If the arrows form a closed path, then there is a cyclic equilibrium, that corresponds to Case 5 above.

In general, for every vector $u \in [1, \infty)^I$, let $I_u = \{i \in I \mid u^i = 1\}$, and

$$V_u = \left\{ v = \sum_{i \in I_u} \alpha_i r_{\{i\}}, \alpha_i \geq 0, \sum_{i \in I_u} \alpha_i = 1, \alpha_i > 0 \Rightarrow v^i = 1 \right\},$$

be all vectors in the convex hull $\text{co}(r_{\{i\}}, i \in I_u)$ such that each player that takes part in the convex combination receives 1. For every $u \in [1, \infty)^3$ draw small arrows in direction $u - v$, for every $v \in V_u$. Any fixed point (that is, if $u \in V_u$), correspond to a stationary ε-equilibrium. Any closed path that is formed by the arrows corresponds to a cyclic ε-equilibrium. More generally, if there is an open path of infinite length then there is a non-cyclic ε-equilibrium. Moreover, if there is an equilibrium payoff in the convex hull of $\{r_{\{i\}}\}_{i \in I}$, then there is either a fixed point, a closed path, or an open path with length infinity.

Alas, as the next example shows, there are 4-player games in which there are no fixed points, and all paths are open and have finite length.

3 The Example

We will study the following four player quitting game:

4

		2			2	
	1	*continue*	$4,1,0,0$	1	$0,0,4,1$	$1,1,0,1$
		$1,4,0,0$	$1,1,1,1$		$1,0,1,1$	$0,1,0,0$
3						
	1	$0,0,1,4$	$0,1,1,1$	1	$1,1,1,1$	$0,0,1,0$
		$1,1,1,0$	$1,0,0,0$		$0,0,0,1$	$-1,-1,-1,-1$

Figure 2

In this game player 1 chooses a row (top row = continue), player 2 chooses a column (left column = continue), player 3 chooses either the top two matrices or the bottom two matrices, (top two matrices = continue) and player 4 chooses either the left two matrices or the right two matrices (left two matrices = continue).

Note that there are the following symmetries in the payoff function: for every 4-tuple of actions (a, b, c, d) we have:

$$\begin{aligned} v^1(a,b,c,d) &= v^2(b,a,d,c), \\ v^1(a,b,c,d) &= v^4(c,d,b,a) \qquad\qquad \text{and} \\ v^2(a,b,c,d) &= v^3(c,d,b,a), \end{aligned}$$

where $v^i(a,b,c,d)$ is the payoff to i if the action combination is (a,b,c,d) $(v^i(c^1,c^2,c^3,c^4) = 0)$.

In section 3.1 we prove that this game admits an equilibrium profile y that has the following structure:

$$y_n = \begin{cases} (x,1,z,1) & n \text{ odd} \\ (1,x,1,z) & n \text{ even} \end{cases}$$

where $x, z \in]0,1[$ are independent of n; that is, at odd stages players 2 and 4 continue, while 1 and 3 quit with positive probability, whereas at even stages 1 and 3 continue, while 2 and 4 quit with positive probability.

Thus, the game admits a cyclic equilibrium with period 2.

We then prove the following:

Proposition 1 *The game does not admit a stationary equilibrium.*

Proposition 2 *For ε small enough, the game does not admit an ε-equilibrium x such that $||x_n - c|| < \varepsilon$ for every n.*

It follows from Propositions 1 and 2 that the game does not admit a stationary ε-equilibrium, provided ε is small enough. Indeed, let us argue by contradiction, and assume that for every ε there exists a stationary ε-equilibrium x_ε. Let $x_\star$ be an accumulation point of $\{x_\varepsilon\}$ as $\varepsilon \to 0$. If $x_\star$ is terminating ($x_* \neq c$) then it is a stationary 0-equilibrium, which is ruled out by Proposition 1. Otherwise, $x_\star = c$, and then, for ε sufficiently small, there is an ε-equilibrium x where $\| x_n - c \| < \varepsilon$, which is ruled out by Proposition 2.

Proposition 1 is proved in section 3.2, while Proposition 2 is proved in section 3.3.

3.1 Cyclic equilibrium

We prove that the game possesses a cyclic equilibrium, where the length of the cycle is 2. At odd stages players 2 and 4 play c^2 and c^4 respectively, and players 1 and 3 continue with probability x and z respectively, both strictly less than 1. At even stages players 1 and 3 play c^1 and c^3 respectively, and players 2 and 4 continue with probability z and x respectively.

Formally, we study now profiles y that satisfy:

$$y_n = \begin{cases} (x, 1, z, 1) & n \text{ odd} \\ (1, z, 1, x) & n \text{ even} \end{cases}$$

where $x, z \in]0, 1[$ are independent of n.

The one-shot game played by players 1 and 3 at odd stages is

		3: z	3: $1-z$
1	x	γ_c^1, γ_c^3	0,1
	$1-x$	1,0	1,1

Figure 3: The game of players 1 and 3 at odd stages

In this game player 1 is the row player, player 3 is the column player, and γ_c^i is the continuation payoff of player $i = 1, 3$. The payoffs received by players 2 and 4 if termination occurs in an odd stage are given by the matrix below, in which the first coordinate of each entry is player 2's payoff, and the second coordinate is player 4's payoff.

	$0, 4$
$4, 0$	$1, 0$

(1)

The one-shot game played by players 2 and 4 at even stages is

		4	
		x	$1-x$
2	z	γ_c^2, γ_c^4	0,1
	$1-z$	$1, 0$	1,1

Figure 4: The game of players 2 and 4 at even stages

where player 2 is the row player, player 4 is the column player, and the payoffs that are received by players 1 and 3 if termination occurs are given by matrix (1). The two situations are identical (up to the continuation payoffs).

We now find necessary conditions on (x, z). First, (x, z) is a fully mixed equilibrium of the matrix game in Figure (3), so that

$$x\gamma_c^3 = 1 \text{ and } z\gamma_c^1 = 1, \tag{2}$$

and both players 1 and 3 receive 1 in this equilibrium.

By the symmetry of the profile, the continuation payoffs (resp. initial payoffs) of players 4 and 2 must coincide with the initial payoffs (resp. continuation payoffs) of players 1 and 3. That is, (γ_c^3, γ_c^1) is the payoff received in the matrix game (1), when the empty entry is filled with $(1, 1)$ and the row and column players play according to x and z respectively, so that

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$$\begin{aligned} \gamma_c^3 &= xz + 4z(1-x) + (1-x)(1-z) \\ \gamma_c^1 &= xz + 4x(1-z) \end{aligned}$$

Using (??), the second equation implies that $x = \frac{1}{z(4-3z)}$, and therefore the first equation translates to $z(4-3z) = 1 + 3z - \frac{1+2z}{z(4-3z)}$, or

$$z(1 - z + 3z^2)(4 - 3z) = 1 + 2z.$$

Call $f(z)$ the polynomial in the left hand side, and $g(z)$ the one on the right hand side. Then $f(0) = 0 < 1 = g(0)$, $f(1) = 3 = g(1)$, and $f'(1) = -1 < 2 = g'(1)$. In particular, there exists $z \in (0, 1)$ such that $f(z) = g(z)$, hence a cyclic equilibrium exists.

3.2 No Stationary Equilibria

We check here that the game has no stationary equilibrium. We organize the discussion according to the number of players who play both actions with positive probability.

It is immediate to check that there is no stationary equilibrium in which at least three players play pure strategies.

We shall now verify that there is no stationary equilibrium where two players play pure stationary strategies. Using the symmetries in the payoff function, it is enough to consider the cases where either player 3 and 4 play pure strategies, or players 2 and 4 play pure strategies.

Assume first that there is an equilibrium in which players 3 and 4 play pure stationary strategies. The strategies of players 1 and 2 form then an equilibrium of a 2×2 game. We will see that these two-player games have only pure equilibria. The four-player game would thus have an equilibrium in pure stationary strategies - a contradiction. In the first three cases, the induced game is equivalent to a one-shot game. In the last case, it is a quitting game.

Case 1: Players 3 and 4 play (q^3, q^4): the unique equilibrium in the induced game is (c^1, c^2).

Case 2: Players 3 and 4 play (c^3, q^4): the unique equilibrium is (c^1, q^2).

Case 3: Players 3 and 4 play (q^3, c^4)— symmetric to case 2.

Case 4: Players 3 and 4 play (c^3, c^4): the unique equilibria are (q^1, c^2) and (c^1, q^2).

We shall now see that there is no stationary equilibrium where players 2 and 4 play pure actions, by analyzing the induced game between players 1 and 3.

Case 1: Players 2 and 4 play (c^2, c^4): the induced game has a unique equilibrium (q^1, q^3).

Case 2: Players 2 and 4 play (q^2, c^4): the unique equilibrium in the induced game is $(\frac{1}{2}c^1 + \frac{1}{2}q^1, \frac{1}{4}c^3 + \frac{3}{4}q^3)$. Player 2 would receive $\frac{5}{8}$, but he would get 1 by playing c^2.

Case 3: Players 2 and 4 play (c^2, q^4): the unique equilibrium is (q^1, c^3).

Case 4: Players 2 and 4 play (q^2, q^4): the unique equilibrium is (c^1, q^3).

Next, we check that there is no stationary equilibrium where one player, say player 4, plays a pure strategy, and all the other players play a fully mixed strategy. We denote by (x, y, z) the fully mixed stationary equilibrium in the three-player game when player 4 plays some pure stationary strategy.

Assume first that player 4 plays q^4. Then, in order to have player 2 indifferent, we should have

$$x(1-z) = z - (1-x)(1-z)$$

which implies that $z = 1/2$. In order to have player 1 indifferent, we should have

$$(1-y)z + y(1-z) = yz - (1-y)(1-z)$$

which solves to $yz = 1/2$, and therefore $y = 1$, which is pure.

Assume now that player 4 plays c^4. First we note that $x < 1/2$, otherwise player 3 prefers to play q^3 over c^3. Next, if player 2 is indifferent between his actions, then

$$\frac{(1-x)(1+3z)}{1-xz} = x + (1-x)z$$

or equivalently,

$$(1-x)(1+2z+xz^2) = (1-xz)x.$$

Since $x < 1/2$, it follows that $1-x > x$. Therefore it follows that

$$1 + 2z + xz^2 < 1 - xz$$

which is clearly false.

3.2.1 No fully mixed stationary equilibrium

We prove now that there is no fully mixed stationary equilibrium. We start by a few notations. Let (x, y, z, t) be a supposed fully mixed stationary equilibrium. By symmetry, we may assume that $y = \min(x, y, z, t)$. Write $a := \gamma^1(x, y, z, t) \geq \gamma^1(0, y, z, t) > 0$.

Assume players 2,3,4 play the stationary profile (y, z, t), and that player 1 plays from the second stage on the stationary profile x. By playing c^1 at stage 1 player 1 gets

$$\alpha(a; y, z, t) := yzt(a-2) - 2yz + 3zt - yt + y + z,$$

whereas by playing q^1 at stage 1 he gets

$$\beta(y, z, t) := t + (1-t)(y + z - 1).$$

Since in a fully mixed equilibrium it is optimal to use both c^1 and q^1 with positive probability, one has

$$a = \beta(y, z, t) = \alpha(a; y, z, t).$$

Therefore, the polynomial

$$\Delta_1(y, z, t) := \alpha(\beta(y, z, t); y, z, t) - \beta(y, z, t)$$

vanishes at (y,z,t). For simplicity, we write

$$\Delta_1(y,z,t) = (\beta - 2)yzt - 2yz + 4zt + 1 - 2t,$$

thereby omitting the arguments in β. We define $\Delta_2(x,z,t), \Delta_3(x,y,t)$ and $\Delta_4(x,y,z)$ in a symmetric way.

To rule out the existence of a fully mixed stationary equilibrium , we shall prove that there is no $(x,y,z,t) \in (0,1)^4$ such that (i) $y = \min(x,y,z,t)$,(ii) $\Delta_1(y,z,t) = \Delta_4(x,y,z) = 0$ and (iii) $\beta(y,z,t) \in (0,1)$.

Lemma 3 *$\Delta_1(t,t,t) > 0$ for each $t \in [0,1]$.*

Proof. $t \mapsto \Delta_1(t,t,t)$ is a polynomial in one variable. The result follows by using any of the standard methods for counting the number of zeroes of a polynomial in a compact interval, e.g. Sturm's method. ■

We list a few useful observations, valid on $\{y \le z, t\}$

Fact 1 β is (strictly) increasing in each variable.

Fact 2 Δ_1 is decreasing in y.

Fact 3 Δ_1 is increasing in z.

The proofs of the first two facts are obtained by elementary algebraic manipulations, and are therefore omitted. For the third one, prove first that $\frac{\partial \Delta_1}{\partial z}$ is decreasing in y. Since $y \le t$, this yields $\frac{\partial \Delta_1}{\partial z}(y,z,t) \ge \frac{\partial \Delta_1}{\partial z}(t,z,t) = (\beta - 2)t^2 + t^2 z(1-t) + 2t > 0$.

By **Fact 2** and **Fact 3**, $\Delta_1(y,z,t) \ge \Delta_1(t,t,t) > 0$ if $y \le t \le z$.

Lemma 4 *The polynomial Δ_1 does not vanish on $\Omega := \{y \le z \le t\} \cap \{z \ge \frac{1}{2}\}$.*

Proof. **Step 1:** $\Delta_1 > 0$ on $\Omega \cap \left\{ y < \frac{1}{2} \right\}$.

Indeed, in that region, one has $y \le \frac{1}{2} \le z \le t$, hence, by **Facts 2** and **3**,

$$\Delta_1(y,z,t) \ge \Delta_1(\frac{1}{2}, \frac{1}{2}, t) = \frac{1}{2} - \frac{t}{2} + \beta \frac{t}{4} > 0.$$

Step 2: $\Delta_1 > 0$ on $\Omega_1 := \{\frac{1}{2} \le y \le z \le t \le \frac{2}{3}\}$.

We restrict (y,z,t) to be in Ω_1. By **Fact 2**, $\Delta_1(y,z,t) \ge \Delta_1(z,z,t)$. We prove below that $\Delta_1(z,z,t)$ is decreasing in z. This will imply $\Delta_1(z,z,t) \ge \Delta_1(t,t,t) > 0$, hence the claim.

Note first that, by **Fact 1**,

$$\beta(y,z,t) \leq \beta(\frac{2}{3},\frac{2}{3},\frac{2}{3}) = \frac{7}{9}. \tag{3}$$

An elementary computation gives

$$\frac{\partial}{\partial t}\frac{\partial}{\partial z}\{\Delta_1(z,z,t)\} = 2z(\beta-2)+2zt(2-y-z)+2z^2(1-t)-2z^2t+4>0.$$

Therefore,

$$\frac{\partial}{\partial z}\Delta_1(z,z,t) \leq \frac{\partial}{\partial z}\Delta_1(z,z,\frac{2}{3}) = \frac{4}{3}z(\beta-2)+\frac{4}{9}z^2-4z+\frac{8}{3}. \tag{4}$$

The right-hand side in (4) is decreasing in z. It is therefore maximal for $z=\frac{1}{2}$. It is then equal to $\frac{2}{3}(\beta-1)+\frac{1}{9}$ which is negative by (3). This proves that $\Delta_1(z,z,t)$ is decreasing in z.

Step 3: $\Delta_1 > 0$ on $\Omega_2 := \{\frac{1}{2} \leq y < \frac{2}{3} \leq z \leq t\}$.

We restrict (y,z,t) to be in Ω_2. Notice first that $\beta \geq \frac{2}{3}$. One has

$$\begin{aligned}\Delta_1(y,z,t) \geq \Delta_1(\frac{2}{3},\frac{2}{3},t) &= (\beta(\frac{2}{3},\frac{2}{3},t)-2)\frac{4}{9}t+\frac{1}{9}+\frac{2}{3}t\\ &\geq \frac{t+1}{9}.\end{aligned}$$

The first inequality uses **Facts 2** and **3**. The second one follows from (**Fact 1**) $\beta(\frac{2}{3},\frac{2}{3},t) \geq \beta(\frac{2}{3},\frac{2}{3},\frac{2}{3}) \geq \frac{3}{4}$. ©

Step 4: $\Delta_1 > 0$ on $\Omega_3 := \{\frac{2}{3} \leq y \leq z \leq t\}$.

We restrict (y,z,t) to be in Ω_3. As in **Step 3**, $\beta \geq \frac{2}{3}$. We prove below that Δ_1 is increasing in t. Using **Fact 2**, this will imply $\Delta_1(y,z,t) \geq \Delta_1(z,z,z) > 0$. One has

$$\frac{\partial \Delta_1}{\partial t} = (\beta-2)yz+yzt(2-y-z)+4z-2 \geq (\beta-2)yz+4z-2. \tag{5}$$

Plainly, the right-hand side of (5) is increasing in z. Therefore, it is minimal when $z=y$, hence at least $-\frac{5}{4}y^2+4y-2$. This latter expression is itself minimized at $y=\frac{2}{3}$ and then equal to $\frac{1}{9}$. Thus, $\frac{\partial \Delta_1}{\partial t} > \frac{1}{9}$.

Step 5: $\Delta_1 > 0$ on $\Omega_4 := \left([\frac{1}{2},\frac{2}{3}] \times [\frac{1}{2},\frac{2}{3}] \times [\frac{2}{3},1[\right) \cap \{y \leq z\}$.

We restrict (y,z,t) to be in Ω_4. By **Fact 1**, $\beta(y,z,t) \geq \frac{2}{3}$. By **Fact 2**, $\Delta_1(y,z,t) \geq \Delta_1(z,z,t)$. These inequalities imply

$$\Delta_1(y,z,t) \geq -\frac{4}{3}z^2t-2z^2+4zt+1-2t. \tag{6}$$

Denote by f the function defined by the right-hand side of (6). We prove below that, for each $t \geq \frac{2}{3}$, the function $f(\cdot, t)$ is positive on $\frac{1}{2}, \frac{2}{3}$. Let $t \geq \frac{2}{3}$ be given. Plainly,

$$f(\frac{1}{2}, t) = \frac{1}{2} - \frac{t}{3} > \frac{1}{6} \text{ and } f(\frac{2}{3}, t) = \frac{2}{27}t + \frac{1}{9} > \frac{1}{7}. \tag{7}$$

Next, note that

$$\frac{\partial f}{\partial z}(z, t) = -\frac{8}{3}zt - 4z + 4t$$

is increasing in t and decreasing in z. Therefore, $\frac{\partial f}{\partial z}(z, t)$ is minimal at $(\frac{2}{3}, \frac{2}{3})$, where it equals $-\frac{32}{27}$, and maximal at $(\frac{1}{2}, 1)$, where it equals $\frac{2}{3}$. In particular

$$\sup_{z \in [1/2, 2/3]} \left| \frac{\partial f}{\partial z}(z, t) \right| \leq \frac{12}{7}. \tag{8}$$

Since each $z \in [\frac{1}{2}, \frac{2}{3}]$ satisfies $|z - \frac{1}{2}| \leq \frac{1}{12}$, or $|z - \frac{2}{3}| \leq \frac{1}{12}$, the inequality $f(z, t) > 0$ follows from (7), (8) and the mean value theorem. ■

Lemma 5 *The polynomial Δ_4 does not vanish on $\Omega_5 := \{y \leq z\} \cap \{z \leq \frac{1}{2}\}$.*

Proof. Recall that we only consider values of (x, y, z, t) such that $\gamma^4(x, y, z, t) > 0$. Therefore

$$\Delta_4(x, y, z) > -2xyz - 2xz + 4xy + 1 - 2y = y(-2xz + 4x - 2) + 1 - 2xz.$$

Denote $f_{x,z}(y) = y(-2xz + 4x - 2) + 1 - 2xz$. Then $f_{x,z}(0) = 1 - 2xz > 0$ and $f_{x,z}(z) = (1 - 2z) + 2xz(1 - z) > 0$. In particular, the right hand side is positive on Ω_5. ■

3.3 Proof of Proposition 2

We first present a sketch of the proof. The proof goes by contradiction. Let $\mathsf{x} = (x_n)$ be an ε-equilibrium such that $\|x_n - c\| < \varepsilon$ for each n. Since each player gets 1 by quitting alone, the probability that no player ever quits is close to zero. Moreover, since x_n is close to c, the quitting coalition is a singleton with high probability. In particular, the sum of the payoffs all four players receive in the ε-equilibrium is close to five. Hence, at least one player gets a payoff strictly higher than 1, while no player receives a payoff that is much below one. We then deduce that for every player i, the probability that i belongs to the quitting coalition is bounded away from zero.

Next, we claim that there is no such ε-equilibrium that gives to players 1 and 2 (or 3 and 4) a payoff higher than one. Indeed, assume such an equilibrium were to exist. In the first stage of the game, both players 1 and 2 would choose to continue with very high probability, since the payoff obtained by quitting is approximately 1. Moreover, they must do so in every stage n such that their expected payoff, starting from stage n, is higher than one, unless the probability that the game reaches stage n is close to zero. Therefore, as long as their continuation payoff exceeds 1 and the probability of surviving is not too small, players 1 and 2 will not contribute to the quitting coalition. However, as long as players 1 and 2 do not contribute, their continuation payoffs increases. Indeed, the expected payoff starting from today is a weighted average of the payoff received if someone quits today and of the expected payoff starting from tomorrow, and the payoff to players 1 and 2 is less than one only if players 3 and 4 quit.

Assume now that player 1 (but not player 2) gets a payoff higher than 1, and let n_1 be the first stage such that the continuation payoff of player 1 is close to one. Since the continuation payoff of player 1 decreases between stages 1 and n_1, the probability that player 2 quits before stage n_1 is non-negligible. Since player 1 hardly contributes to the probability of quitting before stage n_1, the continuation payoffs of player 2 do not decrease over time before stage n_1. Since player 2 quits with non-negligible probability, his continuation payoffs must remain close to one for a while. In particular, players 3 and 4 should not quit in those stages. This implies that the continuation payoffs of player 3 and 4 increase in these stages. After a while (stage n_1 at the latest), both continuation payoffs of players 3 and 4 are higher than one-a situation that has been ruled out above.

To facilitate reading, we let $\rho = 8$ be twice the maximal payoff in absolute value, $\mathcal{N} = \{1, 2, 3, 4\}$ be the set of players, and $N = 4$ be the number of players.

It is convenient to assume that, in any given stage, at most one player quits with positive probability. This assumption entails no loss of generality, as shown by the next lemma.

Lemma 6 *Let $\varepsilon < 1/8$ and let x be an ε-equilibrium such that $\|x_n - c\| < \varepsilon$ for each n. Then there exists a $11N\rho\varepsilon$-equilibrium y such that, for every n, $\|y_n - c\| < \varepsilon$ and $|\{i \in \mathcal{N}, y_n^j > 0\}| \leq 1$.*

Proof. We construct y from x by dividing each stage into four substages, and by letting each player quit in turn with the probability specified by x.

Formally, for $n \in \mathbf{N}$ and $j \in \mathcal{C}$, we set

$$y^i_{(n-1)N+j} = \begin{cases} x^i_n \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}.$$

By construction,

$$\mathbf{P}_{\mathbf{y}}(t > nN | t > (n-1)N) = \mathbf{P}_{\mathbf{x}}(t > n | t > n-1) = \prod_{i=1}^{n} (1 - x^i_n) \text{ for each } n \in \mathbf{N}.$$

Observe next that, for each $j \in \mathcal{N}$,

$$\mathbf{P}_{\mathbf{y}}(S_t = \{i\} | (n-1)N < t \leq nN) = \frac{x^i_n \prod_{j<i}(1-x^j_n)}{1-\prod_{j\in\mathcal{N}}(1-x^j_n)}$$
$$\mathbf{P}_{\mathbf{x}}(S_t = \{i\} | t = n) = \frac{x^i_n \prod_{j\neq i}(1-x^j_n)}{1-\prod_{j\in\mathcal{N}}(1-x^j_n)}$$

The denominator is at least $1 - 4\varepsilon \geq 1/2$, hence the difference between these two probabilities is at most 2ε. Since $\sum_{i\in\mathcal{N}} \mathbf{P}_{\mathbf{y}}(S_t = \{i\} | (n-1)N < t \leq nN) = 1$, one has in particular $\mathbf{P}_{\mathbf{x}}(|S_t| > 1 | t = n) \leq 2N\varepsilon$.

By summation over n, this yields

$$\|\gamma(\mathbf{x}) - \gamma(\mathbf{y})\| \leq 4N\rho\varepsilon. \tag{9}$$

Next, we prove that player i has no pure profitable deviation from $\mathbf{y}^i$.

Consider first the strategy $\mathbf{c}^i$. By repeating the above argument, one has $|\gamma^i(\mathbf{x}^{-i}, \mathbf{c}^i) - \gamma^i(\mathbf{y}^{-i}, \mathbf{c}^i)| \leq 4N\rho\varepsilon$. Since $\mathbf{x}$ is an ε-equilibrium, this yields, by (9),

$$\gamma^i(\mathbf{y}^{-i}, \mathbf{c}^i) \leq \gamma^i(\mathbf{y}) + \varepsilon + 8N\rho\varepsilon.$$

Consider next the strategy $\mathbf{q}^i_{(n-1)N+k}$ that quits in stage $(n-1)N+k$ and continues in the former stages. We compare the payoffs to player i under the two profiles $(\mathbf{y}^{-i}, \mathbf{q}^i_{(n-1)N+k})$ and $(\mathbf{x}^{-i}, \mathbf{q}^i_n)$. By repeating the above argument, one has

$$\left| \mathbf{P}_{\mathbf{y}^{-i},\mathbf{q}^i_{(n-1)N+k}}(t \leq (n-1)N) = \mathbf{P}_{\mathbf{x}^{-i},\mathbf{q}^i_n}(t \leq n-1) \right|$$
$$\left| \mathbf{E}_{\mathbf{y}^{-i},\mathbf{q}^i_{(n-1)N+k}}\left[r^i_{S_t} | t \leq (n-1)N \right] - \mathbf{E}_{\mathbf{x}^{-i},\mathbf{q}^i_n}\left[r^i_{S_t} | t \leq (n-1) \right] \right| \leq 4N\rho\varepsilon.$$

Next, $\mathbf{E}_{\mathbf{y}^{-i},\mathbf{q}^i_{(n-1)N+k}}\left[r^i_{S_t} | t > (n-1)N \right] \leq 1 + (N-1)\rho\varepsilon + \rho\varepsilon$, where $(N-1)\rho\varepsilon$ accounts for the probability that someone may quit in the first $k-1$ substages of stage n, and $\rho\varepsilon$ accounts for the probability that some player other than i

may quit in substage k. Also, $\mathsf{E}_{\mathsf{x}^{-i},\mathsf{q}^i_n}\left[r^i_{S_t}|t > (n-1)\right] \leq 1-\rho N\varepsilon$. Collecting these inequalities yields

$$\gamma^i(\mathsf{y}^{-i}, \mathsf{q}^i_{(n-1)N+k}) \leq \gamma^i(\mathsf{x}^{-i}, \mathsf{q}^i_n) + 6N\rho\varepsilon \leq \gamma^i(\mathsf{y}) + \varepsilon + 10N\rho\varepsilon.$$

This concludes the proof. ■

We henceforth assume that x is an ε-equilibrium such that $|\{i \in \mathcal{N}, x_n > 0\}| \leq 1$ and $\|x_n - c\| < \varepsilon$ for each n. We also take $\varepsilon > 0$ sufficiently small to allow various inequalities to hold.

Lemma 7 *One has*

1. $\mathsf{P}_{\mathsf{x}}(t < +\infty) \geq 1 - \varepsilon$.
2. $\gamma^i(\mathsf{x}) \geq 1 - \rho\varepsilon - \varepsilon$ *for each* $i \in \mathcal{N}$, *and* $\gamma^i(\mathsf{x}) \geq \frac{5}{4} - 2\varepsilon$ *for some* $i \in \mathcal{N}$.
3. $\mathsf{P}_{\mathsf{x}}(S_t = \{i\}) \geq \frac{2}{15} - \rho\varepsilon$ *for each* i.

Proof. Let $\mathsf{y}^{i,n}$ the strategy of player i that coincides with x^i in the first n stage and plays q^i at stage $n+1$. The payoffs $\gamma^i(\mathsf{x}^{-i}, \mathsf{y}^{i,n})$ converge to $\gamma^i(\mathsf{x}) + \mathsf{P}_{\mathsf{x}}(t = +\infty)$, as n goes to infinity. Since $\gamma^i(\mathsf{x}^{-i}, \mathsf{y}^{i,n}) \leq \gamma^i(\mathsf{x}) + \varepsilon$, claim 1 follows.

By quitting in the first stage, player i obtains at least $1 - \rho\varepsilon$. The first part of claim 2 follows. Next, whenever the quitting set is a singleton, the payoffs to the players sum up to 5. Therefore,

$$\sum_{i \in \mathcal{N}} \gamma^i(\mathsf{x}) = 5\mathsf{P}_{\mathsf{x}}(t < +\infty) \geq 5 - 5\varepsilon.$$

In particular, there exists i such that $\gamma^i(\mathsf{x}) \geq \frac{5}{4} - \frac{5}{4}\varepsilon$. The second part of claim 2 follows.

We turn to the proof of 3. For notational convenience, set $p^i := \mathsf{P}_{\mathsf{x}}(S_t = \{i\})$. Note that

$$\gamma^1(\mathsf{x}) = p^1 + 4p^2$$

and that analogous identities hold for players 2, 3 and 4. In particular, by 2, one has

$$p^1 + 4p^2 \geq 1 - 2\rho\varepsilon \text{ and } 4p^1 + p^2 \geq 1 - 2\rho\varepsilon,$$

which implies $p^1 + p^2 \geq \frac{2}{5} - \frac{4}{5}\rho\varepsilon$. By exchanging the roles of the players, one gets $p^3 + p^4 \geq \frac{2}{5} - \rho\varepsilon$. Therefore, $p^1 + p^2 \leq \frac{3}{5} + \rho\varepsilon$. Thus, (p^1, p^2) satisfy

$$p^1 + 4p^2 \geq 1 - 2\rho\varepsilon,\ 4p^1 + p^2 \geq 1 - 2\rho\varepsilon, \text{ and } p^1 + p^2 \leq \frac{3}{5} + \rho\varepsilon. \tag{10}$$

Any solution to the system (10) satisfies $p^1, p^2 \geq \frac{2}{15} - \rho\varepsilon$. ■

Given $i \in \mathcal{N}$, and $n \in \mathsf{N}$, let $\mathsf{x}^i(n)$ be the strategy which plays c^i up to stage n, and coincides with x^i after stage n. Let also $p^i_n := \mathsf{P}_{\mathsf{x}}(t < n, S_t = \{i\})$. Then $p^i = \lim_{n\to\infty} p^i_n$.

Lemma 8 *Assume that $\gamma^i(\mathsf{x}_n) \geq 1 + \sqrt{\varepsilon}$ for some player i and every $n \leq n_0$. Then*

$$\gamma^i(\mathsf{x}^{-i}, \mathsf{x}^i(n)) \geq \gamma^i(\mathsf{x}) + \sqrt{\varepsilon} p^i_n, \text{ for every } n \leq n_0.$$

Proof. We proced by induction. Assume $n = 1$. If $x^i_1 = 0$, then $\mathsf{x}^i(1) = \mathsf{x}^i$ and $p^i_1 = 0$, and the result holds. Otherwise, $p^i_1 = 1 - x^i_1$, hence

$$\gamma^i(\mathsf{x}) = p^i_1 + (1 - p^i_1)\gamma^i(\mathsf{x}^{-i}, \mathsf{x}^i(1)).$$

Then

$$\gamma^i(\mathsf{x}^{-i}, \mathsf{x}^i(1)) = \gamma^i(\mathsf{x}) + \frac{p^i_1}{1 - p^i_1}(\gamma^i(\mathsf{x}) - 1) \geq \gamma^i(\mathsf{x}) + \sqrt{\varepsilon} p^i_1.$$

Assume now that $1 < n \leq n_0$. If $x^i_n = 0$, then $\mathsf{x}^i(n) = \mathsf{x}^i(n-1)$ and $p^i_n = p^i_{n-1}$. In particular, by the induction hypothesis,

$$\gamma^i(\mathsf{x}^{-i}, \mathsf{x}^i(n)) = \gamma^i(\mathsf{x}^{-i}, \mathsf{x}^i(n-1)) \geq \gamma^i(\mathsf{x}) + \sqrt{\varepsilon} p^i_{n-1} = \gamma^i(\mathsf{x}) + \sqrt{\varepsilon} p^i_n,$$

and the result holds.

If $x^i_n > 0$ then, applying the case $n = 1$ to the profile x_{n-1} we get

$$\gamma^i(\mathsf{x}^{-i}_{n-1}, \mathsf{x}^i(n)_{n-1}) \geq \gamma^i(\mathsf{x}^{-i}_{n-1}, \mathsf{x}^i(n-1)_{n-1}) + \sqrt{\varepsilon}(1 - x^i_n).$$

Using the induction hypothesis we get:

$$\begin{aligned}
\gamma^i(\mathsf{x}^{-i}, \mathsf{x}^i(n)) &\geq \gamma^i(\mathsf{x}^{-i}, \mathsf{x}^i(n-1)) + \mathsf{P}_{\mathsf{x}^{-i},\mathsf{c}^i}(t \geq n-1)\sqrt{\varepsilon}(1 - x^i_n) \\
&\geq \gamma^i(\mathsf{x}) + \sqrt{\varepsilon}(p^i_{n-1} + \mathsf{P}_{\mathsf{x}^{-i},\mathsf{c}^i}(t \geq n-1)(1 - x^i_n)) \\
&\geq \gamma^i(\mathsf{x}) + \sqrt{\varepsilon} p^i_n.
\end{aligned}$$

■

We say that players 1 and 2 (resp. 3 and 4) are *partners*. The partner of player i is denoted by $\tilde{i}$.

Lemma 9 *Let $a, b > 0$ and let $\varepsilon > 0$ be sufficiently small. Let* y *be a $b\varepsilon$-equilibrium such that $||y_n - c|| < \varepsilon$ for each n. Let $i \in \mathcal{N}$, and assume that $\gamma^i(\mathsf{y}) \geq 1 + a$. Then there exists n_1 such that (i) $\gamma^i(\mathsf{y}_{n_1}) < 1 + \sqrt{\varepsilon}$, (ii) $p^i_{n_1} \leq (b+1)\sqrt{\varepsilon}$, and (iii) $a \leq 3p^{\tilde{i}}_{n_1} + \sqrt{\varepsilon}$.*

Proof. For convenience, assume $i = 1$. Since $p^1 \geq 2/15 - 3\varepsilon$, Lemma 8 implies that there exists a stage n such that $\gamma^1(\mathsf{y}_n) < 1 + \sqrt{\varepsilon}$. Let n_1 be the first such stage. In particular, (i) holds. Observe that $\gamma^1(\mathsf{y}_{n_1-1}) \geq 1 + \sqrt{\varepsilon}$, hence by Lemma 8 $b\varepsilon \geq \sqrt{\varepsilon} p^i_{n_1-1}$. Since the probability that player 1 quits in stage $n_1 - 1$ is at most ε, (ii) follows.

We now prove (iii). Since $\gamma^1(\mathsf{y}_{n_1}) < 1 + \sqrt{\varepsilon}$ one has

$$\begin{aligned} 1 + a \leq \gamma^1(\mathsf{y}) &= p^1_{n_1} + 4p^2_{n_1} + (1 - p^1_{n_1} - p^2_{n_1} - p^3_{n_1} - p^4_{n_1})\gamma^1(\mathsf{y}_{n_1}) \\ &\leq p^1_{n_1} + 4p^2_{n_1} + (1 - p^1_{n_1} - p^2_{n_1}) + \sqrt{\varepsilon} \\ &\leq 1 + 3p^2_{n_1} + \sqrt{\varepsilon}, \end{aligned}$$

and (iii) follows. ■

Corollary 10 *Let $b > 0$ and $a > 3(b+2)\sqrt{\varepsilon}$. There is no $b\varepsilon$-equilibrium* y *such that :*

- *$||y_n - c|| < \varepsilon$ for each n*
- *$\gamma^i(\mathsf{y}), \gamma^{\tilde{i}}(\mathsf{y}) \geq 1 + a$ for some $i \in \mathcal{N}$.*

Proof. Let y be such a $b\varepsilon$-equilibrium, and assume w.l.o.g. $i = 1$. Apply Lemma 9 twice, to players 1 and 2. Call n_1 and n_2 the corresponding two stages, and assume w.l.o.g that $n_1 \leq n_2$. Thus, one has both $p^2_{n_1} \geq a/3 - \sqrt{\varepsilon}/3$, and $p^2_{n_2} \leq (b+1)\sqrt{\varepsilon}$. Since $n_1 \leq n_2$ $p^2_{n_1} \leq p^2_{n_2}$. Thus $a - \sqrt{\varepsilon} \leq 3(b+1)\sqrt{\varepsilon}$ — a contradiction. ■

End of proof of Proposition 2: We assume w.l.o.g. that $\gamma^1(\mathsf{x}) \geq 5/4 - 2\varepsilon$. We will exhibit a stage n_2 such that x_{n_2} is a 8ε-equilibrium, and $\gamma^3(\mathsf{x}_{n_2}), \gamma^4(\mathsf{x}_{n_2}) \geq 1 + 1/12$. By Corollary 10, we get a contradiction.

Apply Lemma 9 to x and $i = 1$, and denote n_1 the corresponding stage. Thus, $p^1_{n_1} \leq 2\sqrt{\varepsilon}$ and $p^2_{n_1} \geq \frac{1}{3} \times \frac{1}{4} - \sqrt{\varepsilon}$. By Lemma 8, there exists a stage $N_2 < n_1$ with $\gamma^2(\mathsf{x}_{N_2}) < 1 + \sqrt{\varepsilon}$ (otherwise, $\gamma^2(\mathsf{x}_{n_1} \geq \gamma^2(\mathsf{x}) + \frac{2}{3}\sqrt{\varepsilon} - \varepsilon/3$, contradicting the fact that x is an ε-equilibrium). We set

$$n_2 = \max\{n \leq n_1, \gamma^2(\mathsf{x}_n) \leq 1 + \sqrt{\varepsilon}\}.$$

Since $p^1_{n_2} \leq p^1_{n_1} \leq 2\sqrt{\varepsilon}$, and $p^1 \geq \frac{2}{15} - \rho\varepsilon$, one has

$$\mathsf{P}_{\mathsf{x}}(t < n_2) \leq 1 - \mathsf{P}(t \geq n_2 \text{ and } S_t = \{1\}) \leq \frac{13}{15} + \rho\varepsilon + 2\sqrt{\varepsilon} \leq \frac{7}{8}.$$

Since x is an ε-equilibrium, x_{n_2} is a 8ε-equilibrium.

Our next goal is to prove that $p^2_{n_2} \geq \frac{1}{12} - 17\sqrt{\varepsilon}$. If $n_2 = n_1$ there is nothing to prove. Assume $n_2 < n_1$. This means that $\gamma^2(\mathsf{x}_{n_1}) > 1 + \sqrt{\varepsilon}$. By the definition of n_2, $\gamma^2(\mathsf{x}_k) > 1 + \sqrt{\varepsilon}$ for every $n_2 < k \leq n_1$. Apply Lemma 8 with $\mathsf{y} = \mathsf{x}_{n_2}$ (thus $y_k = x_{n_2+k}$, for each k) and $n = n_1 - n_2$. Since x_{n_2} is a 8ε-equilibrium, the conclusion, rephrased in terms of x, is that $\mathsf{P}_{\mathsf{x}}(t < n_1, S_t = \{2\}\,|t \geq n_2) \leq 8\varepsilon/\sqrt{\varepsilon} = 8\sqrt{\varepsilon}$. In particular $p^2_{n_1} - p^2_{n_2} \leq 8\sqrt{\varepsilon}$, and therefore $p^2_{n_2} \geq \frac{1}{12} - 9\sqrt{\varepsilon}$.

We use this result to prove that $\gamma^3(\mathsf{x}_{n_2}), \gamma^4(\mathsf{x}_{n_2}) \geq 1 + 1/12$.

As previously, one has

$$1 - 2\rho\varepsilon \leq \gamma^2(\mathsf{x}) = 4p^1_{n_2} + p^2_{n_2} + \left(1 - \sum_{i \in \mathcal{N}} p^i_{n_2}\right) \gamma^2(\mathsf{x}_{n_2}). \tag{11}$$

By definition of n_2, $\gamma^2(\mathsf{x}_{n_2}) \leq 1 + \sqrt{\varepsilon}$. Since $p^1_{n_2} \leq p^1_{n_1} \leq 2\sqrt{\varepsilon}$, one deduces from (11) that $p^3_{n_2} + p^4_{n_2} \leq (7 + 2\rho\varepsilon)\sqrt{\varepsilon} + 4\rho\varepsilon \leq 8\sqrt{\varepsilon}$.

On the other hand,

$$1 - 2\rho\varepsilon \leq \gamma^3(\mathsf{x}) = 4p^4_{n_2} + p^3_{n_2} + \left(1 - \sum_{i} p^i_{n_2}\right) \gamma^3(\mathsf{x}_{n_2}). \tag{12}$$

Since $p^2_{n_2} \geq 1/12 - 17\sqrt{\varepsilon}$, (12) yields $\gamma^3(\mathsf{x}_{n_2}) \geq 1 + \frac{1}{11} - \varepsilon^{1/4} \geq 1 + 1/12$. Similarly, $\gamma^4(\mathsf{x}_{n_2}) \geq 1 + \frac{1}{12}$. Since x_{n_2} is a 8ε-equilibrium, we get a contradiction to Lemma 10. ■

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